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GEOMETRIC SHAPES OF MEMBRANE SHELLS STIFFENED WITH AN ELASTIC RING

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A nonlinear integro-differential equation for determination of the shape of the meridian of a shell of revolution for which a prescribed external load produces no change in the curvature and torsion of the median surface has been derived. A formula for determination of the cross-sectional area of an elastic stiffening ring has been obtained.

Consideration is given to an elastic shell of revolution which is bounded by the planes perpendicular to the axis of rotation and which is deformed by an axisymmetric load acting on the surface and at the edges of the shell. It is necessary to determine the shape of the meridian and of a bearing ring for which a prescribed external load generates no couple stresses in the shell.

According to the membrane theory of shells, determination of forces in the shell of revolution loaded symmetrically about the axis of rotation is reduced to solution of the system of equations [1, 2]

$$\frac{d}{d\varphi}(rN_\varphi) - r_1 N_\theta \cos\varphi + q_\varphi r r_1 = 0\,, \quad rN_\varphi + r_1 N_\theta \sin\varphi + q_n r r_1 = 0\,. \tag{1}$$

After obvious transformations, system (1) takes the form

$$\frac{d}{d\varphi}(r\sin\varphi N_\varphi) + (q_n\cos\varphi + q_\varphi \sin\varphi)\, r r_1 = 0\,, \quad \frac{N_\varphi}{r_1} + \frac{N_\theta}{r_2} + q_n = 0\,. \tag{2}$$

The radii of curvature of the meridian r_1 and the median surface r_2 in the plane which is perpendicular to the meridian are related to the radius of the shell r (of the parallel circle) and the angle φ formed by the normal $\mathbf{n}$ to the median surface and the axis of rotation by the following formulas [1, 2]:

$$\frac{1}{r_1} = \frac{d\sin\varphi}{dr}\,, \quad \frac{1}{r_2} = \frac{\sin\varphi}{r}\,. \tag{3}$$

Introducing the notation

$$\eta = \operatorname{cosec}\varphi\,, \tag{4}$$

we obtain the solution of system (2) in the form

$$N_\varphi = \frac{1}{r}\left[C - \int_{r_0}^{r}\left(q_n + \frac{q_\varphi}{\sqrt{\eta^2-1}}\right) r dr\right]\eta\,, \quad N_\theta = -\,q_n r\eta + \left[C - \int_{r_0}^{r}\left(q_n + \frac{q_\varphi}{\sqrt{\eta^2-1}}\right) r dr\right]\frac{d\eta}{dr}\,, \tag{5}$$

where C is the integration constant and r_0 is the radius of the initial circle of the shell of revolution. If this edge of the shell is loaded by a load (uniformly distributed along the parallel) with an intensity of q = const in parallel to the axis of rotation, from the first equation of (5) we have [2]

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$$C = -r_0 q \,. \tag{6}$$

If $r_0 = 0$ (no opening), then $q = 0$ and $C = 0$.

To derive the resolving equation of the problem we use the equation of compatibility (continuity) of deformations, which in the case of linear deformations and isotropic bodies in question has the form [2]

$$\frac{d}{dr}(r\varepsilon_\theta) = \varepsilon_\varphi \,. \tag{7}$$

Transforming (7) with the use of Hooke's law

$$\varepsilon_\theta = \frac{1}{Eh}(N_\theta - \nu N_\varphi)\,, \quad \varepsilon_\varphi = \frac{1}{Eh}(N_\varphi - \nu N_\theta)\,, \tag{8}$$

from (7) and (8) for E and ν = const and $h = h(r)$ we find

$$N_\varphi - \nu N_\theta = h\frac{d}{dr}\left[\frac{r}{h}(N_\theta - \nu N_\varphi)\right]. \tag{9}$$

With account for (5), Eq. (9) yields the nonlinear integro-differential equation of second order for $\eta = \csc\varphi$

$$\begin{aligned} &r^2\left[C - \int\limits_{r_0}^{r}\left(q_n + \frac{q_\varphi}{\sqrt{\eta^2-1}}\right) r dr\right]\frac{d^2\eta}{dr^2} + r\left[-2q_n r^2 - \frac{q_\varphi r^2}{\sqrt{\eta^2-1}} + \frac{h - h'r}{h}C - \frac{h-h'r}{h}\int\limits_{r_0}^{r}\left(q_n + \frac{q_\varphi}{\sqrt{\eta^2-1}}\right) r dr\right]\frac{d\eta}{dr} + \\ &+\left[r^3\frac{dq_n}{dr} - \frac{2h - h'r}{h}q_n r^2 + \frac{\nu q_\varphi r^2}{\sqrt{\eta^2-1}} - \frac{h-h'r\nu}{h}C + \frac{h-h'r}{h}\int\limits_{r_0}^{r}\left(q_n + \frac{q_\varphi}{\sqrt{\eta^2-1}}\right) r dr\right]\eta = 0\,, \end{aligned} \tag{10}$$

where $h' = dh/dr$. The sought shape of the meridian $r = r(x)$ is determined from the equation

$$\frac{dr}{dx} = \sqrt{\eta^2 - 1}\,. \tag{11}$$

Since the solution of Eq. (10) is a function of r, i.e., $\eta = \eta(r)$, Eq. (11) allows separation of variables

$$x = x_0 + \int\limits_{r_0}^{r}\frac{dr}{\sqrt{\eta^2-1}}\,, \tag{12}$$

where (x_0, r_0) are the coordinates of the starting point of the meridian.

Another variant of the resolving system of equations can be obtained when equilibrium conditions are employed for the part of the shell lying above the parallel circle. In this case, the initial system of equations is written as follows [2]:

$$2\pi r N_\varphi \sin\varphi + R = 0\,, \quad N_\varphi r + r_1 \sin\varphi N_\theta + q_n r r_1 = 0\,. \tag{13}$$

Here R is the projection of the resultant vector of the total load applied to the above-mentioned part of the shell onto the axis of rotation.

The solution of the system of equations (13) has the form

$$N_\varphi = -\frac{R\eta}{2\pi r}\,,$$

$$N_\theta = -\frac{\eta}{r_1}\left[q_n r r_1 - \frac{2R\eta}{2\pi}\right]. \tag{14}$$

From (14) and (9) we obtain the equation for determination of $\eta = \eta(r)$. The meridian shape sought is found upon substitution of this function into (12).

Formulas (11)–(14) ensure a nonbending shape of the meridian and, naturally, nonbending stressed state of the shell in its main part since bending stresses can appear in the vicinity of the boundary. We can circumvent this by stiffering the shell with circular rings with suitable rigidities. The parameters of such a ring are determined from the conditions of equality of the circular deformations of the ring and the shell and those of equilibrium of the ring element:

$$\varepsilon'_\theta = \varepsilon_\theta\ ; \quad S\sigma'_\theta \pm h\sigma_\varphi r \cos\varphi = 0\ . \tag{15}$$

Here ε'_θ and δ'_θ are the deformation and stress in the ring and S is the area of its cross section. For the sake of simplicity we will assume that the shell and the ring are manufactured from the same material. Then, by virtue of Hooke's law, (15) takes the following form:

$$\sigma'_\theta = \sigma_\theta - \nu\sigma_\varphi\ ; \quad S\sigma'_\theta \pm h\sigma_\varphi r \cos\varphi = 0\ ,$$

whence we obtain

$$S = \mp \frac{h\sigma_\varphi r \cos\varphi}{\sigma_\theta - \nu\sigma_\varphi}\ , \tag{16}$$

or

$$S = \mp \frac{hN_\varphi r \cos\varphi}{N_\theta - N\sigma_\varphi}\ .$$

Here the + or − sign is selected from the condition $S > 0$.

In the case of a negative value of the fraction on the right-hand side of expression (16) it can be applied to determination of the cross-sectional area of the external boundary ring, whereas in the case of a positive value it can be applied to determination of the area of the internal ring.

NOTATION

x, distance along the axis of rotation of the shell; r, radius of the shell; h, thickness; φ, angle between the normal to the shell and the axis of rotation; r_1 and r_2, radii of curvature of the meridian and the cross section in the plane perpendicular to the meridian; θ, polar angle; ε_φ and ε_θ and N_φ, N_θ, σ_φ, and σ_θ, deformations and forces in the meridian plane and in the plane perpendicular to the meridian; E and ν, Young modulus and Poisson coefficient; q_n and q_φ, normal and tangential components of the load.

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